Probability theory

Readings

Quiz

Basic concepts

Conditional expectations

Readings

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Probability theory is covered in Appendix B.

- Verify that you know the contents of Appendix B, but skip:
  - in B.3: "Skewness and kurtosis"
  - in B.1: "Continuous random variables" (will be covered next week)
  - B.5: "The Normal and Related distributions" (Normal will be covered next week, so consider it part of next week's reading)

Quiz

Outline

- Goal:
  - Review probability theory concepts
    - as quickly as possible
    - focus on discrete random variables
  - Focus on conditional expectations
- Strategy:
  - Solve problems to check/update your understanding

Problem 1

You are going to throw two dice. Let X_1 be the number of eyes showing on the first die, and X_2 be the number of eyes on the second die. The random variable Y to be the sum of the dice: $Y = X_1 + X_2$

1. Write down the sample space of Y
2. Write down the conditional pdf of Y given $X_1 \leq 2$
3. What is the marginal cdf of Y? Give the function, and draw a picture.
4. Find $E[X_2|Y = y]$
5. Find $Cov\,(X_1, Y)$

Problem 2

[update for 2017]

Basic concepts

Random variables

We will work with two discrete **random variables** X and Y, with **outcomes** $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_m\}$

Joint distribution

The **joint distribution** is given by

$$f_{x,y}(x, y) \equiv P(X = x, Y = y).$$

Think of it as a table,

y/x	x_1	x_2	$\cdots$	x_k
y_1	$f_{x,y}(x_1, y_1)$	$f_{x,y}(x_2, y_1)$		$f_{x,y}(x_k, y_1)$
y_2	$f_{x,y}(x_1, y_2)$	$f_{x,y}(x_2, y_2)$		$f_{x,y}(x_k, y_2)$
$\vdots$				
y_m	$f_{x,y}(x_1, y_m)$	$f_{x,y}(x_2, y_m)$		$f_{x,y}(x_k, y_m)$

Marginal distribution

If we do not know anything about Y, and are only interested in X, then we work with the **marginal distribution**

$$f_x(x) \equiv P(X = x) \equiv \sum_y f_{x,y}(x, y)$$

where the sum is over all the outcomes y. The marginal distribution for Y, $f_y(y)$, is defined similarly.

Conditional distribution

If we know that $Y = y_k$, we should take that into account for assigning probabilities to outcomes of X.

The **conditional probability distribution** is

$$f_{x|y}(x|y) \equiv P(X = x|Y = y) \equiv \frac{f_{x,y}(x, y)}{f_y(y)}$$

Independence

Two random variables X and Y with outcomes $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_m\}$ are **independent** if and only if

$$f_{x,y}(x, y) = f_x(x) f_y(y) \text{ for all outcomes } x \text{ and } y.$$

Independence: conditional version

If the random variables X and Y are independent, then

$$\begin{aligned} f_{x|y}(x|y) &= \frac{f_{x,y}(x,y)}{f_y(y)} \\ &= \frac{f_x(x)f_y(y)}{f_y(y)} \\ &= f_x(x). \end{aligned}$$

Information about Y does not affect the probability with which we see outcomes of X.

Conditional expectations

Definition

The conditional expectation of Y given $X = x$ is

$$E(Y|X = x) = \sum_{y} y f_{y|x}(y|x).$$

This definition is similar to that of the *unconditional* expectation $E(Y)$, but uses the *conditional* probability distribution instead of the *marginal*.

Definition (2)

Sometimes we will use the object $E(Y|X)$. This object differs from $E(Y|X = x)$.

Importantly, $E(Y|X)$ is a random variable.

Importance

Conditional expectations are the main object of interest for modern work in applied econometrics.

- There are settings in which $\Delta = E(Y|X = x') - E(Y|X = x)$ can be interpreted as the causal effect of X on Y
- It is interesting to think about settings in which this is not the case;
- It is then interesting to formulate other conditional expectations that **do** admit that interpretation
- In BUEC 333, the course revolves around $E(Y|X) = \beta_0 + \beta_1 X$
  - Why the linearity?

CE.1

The first property (CE.1) of conditional expectations says that, for any random variable X, and for any function $c : \{x_1, \cdots, x_k\} \to \mathbb{R}$,

$$E(c(X)|X) = c(X)$$

- What does this mean?
- Is it a random variable?

CE.2 - Linearity

The second property states that, for any two random variables X and Y, and for any functions $a, b : \{x_1, \cdots, x_k\} \to \mathbb{R}$,

$$E(a(X)Y + b(X)|X) = a(X)E(Y|X) + b(X).$$

Just like the unconditional expectation, the conditional expectation operator is linear.

CE.3 - Independence

Let X and Y be two independent random variables. Then $E(Y|X) = E(Y)$.

LIE

The *L*aw of *I*terated *E*xpectations (LIE) says that, for any two random variables X and Y,

$$E(E(Y|X)) = E(Y).$$

Proof: [blackboard]

A more advanced version states that, for any three random variables X, Y, Z,

$$E(Y|X) = E(E(Y|X, Z)).$$

Notions of unrelatedness

1. Remember: $X \perp Y \Rightarrow E(Y|X) = E(Y)$
2. CE.5: $E(Y|X) = E(Y) \Rightarrow Cov(X, Y)$

Independence is stronger than mean-independence is stronger than uncorrelatedness.

The reverse is not true!

Table 2: $Y = X^2$

y/x	-1	0	1
0	0	1/3	0
1	1/3	0	1/3

- What is $Cov(X, Y)$?
- What is $E(Y|X)$?

Notes on the notions

1. If u and X are random variables and we assume that $E(u|X) = 0$, then:
    - $E(u) = 0$
    - $Cov(X, u) = 0$
2. If X, Y follow a Normal distribution (?), all notions are equal